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Stochastic resonance in the Landau–Ginzburg equation

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Abstract. The mechanism of stochastic resonance is studied in the case of the Landau-Ginzburg equation stochastically and periodically perturbed, by taking advantage of recent developments on the stochastic partial differential equations. Analytical expressions are given for computing the exit times of the system and to estimate the range of the noise for which the stochastic resonance is possible.

1. Introduction

During the last few years there has been a growing interest in the theory and applications of stochastic differential equations. Most of the known results concern systems of ordinary differential equations (ODE) stochastically perturbed. For these cases the theory by Ventsel and Freidlin (1970) can be applied. Recently an important task has been achieved by extending this theory to partial differential equations (PDE) by Faris and Jona-Lasinio (1982, hereafter referred to as FJ). In FJ a rigorous generalisation of the Ventsel and Freidlin theory has been proved for the nonlinear Landau-Ginzburg equation:

$$\partial_t \phi = m\phi - \phi^3 + A\partial_x^2 \phi + \sqrt{\varepsilon}\eta(x, t) \tag{1}$$

where $x \in [0, L]$ and $\eta(x, t)$ is a white noise δ-correlated both in x and t.

In this paper we discuss the statistical properties of equation (1) subject to a periodic perturbation:

$$\partial_t \phi = m\phi - \phi^3 + A\partial_x^2 \phi + B(x)\cos\omega t + \sqrt{\varepsilon}\eta(x, t). \tag{2}$$

In the case of stochastically perturbed ODE the effect of a periodic forcing can produce the phenomenon of stochastic resonance described theoretically by various authors (Benzi *et al* 1981, 1983, Nicolis 1982, Eckman and Thomas 1982) and experimentally by Fauve and Heshot (1983). By stochastic resonance we mean the case of a dynamical system subject to both periodic and stochastic forcing which show a resonance (peak in the power spectrum) which is absent when either the forcing or the periodic perturbation is absent. Our aim is to investigate the mechanism of stochastic resonance for equation (2).

The paper is organised as follows: in § 2 we discuss the statistical properties of equation (1) using analytical estimates and the results obtained in FJ; in § 3 we discuss the mechanism of stochastic resonance for equation (2) together with some numerical computations.

2. Analytical estimate

It was shown by Ventsel amd Freidlin (1970) that a system of ODE stochastically perturbed is equivalent, in the limit of small variance of the noise, to a Markov chain whose state are the steady states and the periodic orbits of the deterministic equation. Similar conclusions hold for the nonlinear partial differential equation (1). It is known that the deterministic part of equation (1) has no periodic solution, its steady states being the extrema of the functional:

$$V[\phi(x)] = \int \mathrm{d}x[\tfrac{1}{4}\phi^4 - \tfrac{1}{2}m\phi^2 + \tfrac{1}{2}A(\partial_x\phi)^2]. \tag{3}$$

Using equation (4), (1) can be written in the form:

$$\partial_t\phi = -\delta V/\delta\phi + \sqrt{\varepsilon}\eta(x, t). \tag{4}$$

The minima and maxima of V correspond respectively to stable and unstable steady state solutions of equation (4) for $\varepsilon = 0$. Obviously the choice of the boundary conditions determine the x dependence of the steady states. In FJ Dirichlet's boundary conditions have been used: $\phi(0, t) = \phi(L, t) = 0$. In this paper we choose Neuman's boundary conditions: $\partial_x\phi(0, t) = \partial_x\phi(L, t) = 0$. We remark that our results can be straightforwardly generalised to Dirichlet's boundary conditions. We motivated our choice because of future application of equations like (2) to climate dynamics.

The solutions of the nonlinear differential equations:

$$m\phi - \phi^3 + A\,\mathrm{d}^2\phi/\mathrm{d}x^2 = 0, \qquad \mathrm{d}\phi(0)/\mathrm{d}x = \mathrm{d}\phi(L)/\mathrm{d}x = 0 \tag{5a, b}$$

are the extrema of V with Neuman's boundary conditions. Equation (5a) has three trivial solutions, namely

$$\phi_0 = 0, \qquad \phi_+ = m^{1/2}, \qquad \phi_- = -m^{1/2}. \tag{6}$$

It is easy to show that ϕ_0 is an unstable steady state and $\phi_{+,-}$ are two stable steady states for the deterministic part of equation (4). Moreover

$$V(\phi_+) = V(\phi_-) \equiv -Lm^2/4 \equiv V_s. \tag{7}$$

For sufficiently small value of the parameter $R \equiv A/mL^2$, a class of unstable steady states exists, called 'instantons' or 'multi-instantons' solutions. Figure 1 shows the one-instanton solution. We shall indicate by $\phi^{(k)}$ a k-instanton solution, i.e. a solution of equations (5) which crosses the axis $\phi = 0$ k times. It is possible to show that

$$V(\phi^{(1)}) < V(\phi^{(2)}) < V(\phi^{(3)}) \ldots. \tag{8}$$

By the generalisation of the Ventsel and Freidlin's theory obtained in FJ, we know that the statistical properties of equation (1) depend mostly on the average exit times τ_+, τ_- from the basins of attraction of the stable steady states ϕ_+ and ϕ_- respectively. Because of the symmetry in the problem

$$\tau_+ = \tau_- \equiv \tau_s. \tag{9}$$

A direct consequence of the Ventsel and Freidlin's theory is that:

$$\tau = C\exp(2\Delta V/\varepsilon) \tag{10}$$

where $\Delta V = \min\{V(\phi_0) - V_s,\ V(\phi^{(1)}) - V_s,\ V(\phi^{(2)}) - V_s, \ldots\}$ and C is a constant

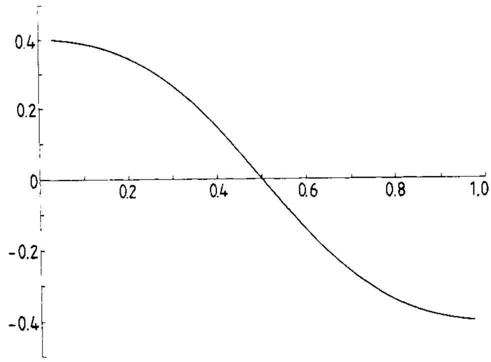

Figure 1. The one-instanton solution, as defined in the text, obtained by a numerical computation of equation (5), with $L=1$, $m=0.25$ and $A=0.0125$.

independent of ε. For sufficiently large value of R $V(\phi^{(1)}) < V(\phi_0)$. Thus by virtue of (8) it follows that

$$\Delta V = V(\phi^{(1)}) - V_s. \tag{11}$$

We have investigated numerically the validity of equation (10) using a discretised version of equation (1). We introduced a regular lattice of spacing $\Delta x = L/N$, and indicate by Ψ_i the scalar field ϕ computed in $i\Delta x$ where $i=0, 1, \ldots, N$. Taking into account the boundary conditions, the discretised version of equation (1) is equivalent to the set of stochastic differential equations

$$\begin{aligned} &\mathrm{d}\Psi_0 = [m\Psi_0 - \Psi_0^3 + A(\Psi_1 - \Psi_0)/\Delta x^2]\,\mathrm{d}t + (\varepsilon/\Delta x)^{1/2}\,\mathrm{d}W_0 \\ &\mathrm{d}\Psi_j = [m\Psi_j - \Psi_j^3 + A(\Psi_{j+1} - 2\Psi_j + \Psi_{j-1})/\Delta x^2]\,\mathrm{d}t + (\varepsilon/\Delta x)^{1/2}\,\mathrm{d}W_j \qquad j=1,\ldots,N-1 \\ &\mathrm{d}\Psi_N = [m\Psi_N - \Psi_N^3 + A(\Psi_{N-1} - \Psi_N)/\Delta x^2]\,\mathrm{d}t + (\varepsilon/\Delta x)^{1/2}\,\mathrm{d}W_N. \end{aligned} \tag{12}$$

Equations (12) can also be written as

$$\mathrm{d}\Psi_k = (-\partial V_N/\partial \Psi_k)(1/\Delta x)\,\mathrm{d}t + (\varepsilon/\Delta x)^{1/2}\,\mathrm{d}W_k, \tag{13}$$

where

$$V_N = [\Sigma_j \Psi_j^4/4 - m/2\Sigma_j \Psi_j^2 + A/2\Delta x^2 \Sigma_j (\Psi_{j+1} - \Psi_j)^2]\Delta x. \tag{14}$$

In analogy with equation (1), the quantities to be computed are the exit times τ_{SN} from the stable steady states $\Psi_{\pm i} = \pm m^{1/2}$, $i=0, 1, \ldots, N$. Using the ray method (Ludwig 1975, Shuss 1980) we obtain

$$\tau_{SN} = C_N \exp(2\Delta V_N/\varepsilon)$$

where

$$\begin{aligned} &\Delta V_N = V_N^{(1)} - V_{SN} \\ &\Delta V_{SN} = -(N+1)m^2\Delta x/4 \\ &V_N^{(1)} = [\Sigma_j \Psi_j^{(1)^4}/4 - m/2\Sigma_j \Psi_j^{(1)^2} + A/(2\Delta x^2)\Sigma_j(\Psi_{j+1}^{(1)} - \Psi_j^{(1)})^2]\Delta x \\ &\Psi_j^{(1)} = \phi^{(1)}(j\Delta x). \end{aligned}$$

The quantity C_N can be calculated explicitly using the eigenvalues of the matrix $L_{ij} \equiv \partial^2 V_N / \partial\Psi_i \, \partial\Psi_j$ computed at the stable and unstable steady states. We refer to appendix 1 for a detailed explanation of this method. We assume that:

$$C = \lim_{N\to\infty} C_N \tag{15}$$

$$\Delta V = \lim_{N\to\infty} \Delta V_N. \tag{16}$$

In table 1 the results are reported of a series of numerical computations to obtain C_N and ΔV_N for increasing values of N. We see that the numerical results have only a few per cent of fluctuations around a given value which we assume to be the asymptotic limit for C and ΔV. By using equation (10) we can compute τ_s as a function of ε.

Table 1.

N	C_N	ΔV_N
20	1.594 03	0.012 87
30	1.575 21	0.012 87
40	1.568 47	0.012 88
50	1.591 16	0.012 88
60	1.565 65	0.012 88
70	1.572 04	0.012 88
80	1.587 46	0.012 88

We also performed a series of numerical tests for equation (10) using the discretised version (12) of equation (1) for various values of ε and $N = 20$, 30, 40 and 50. We have employed the Heun numerical method (see Blum 1972) for stochastic differential equations. All the numerical results agree within a few per cent with the theoretical estimate given by equation (10).

3. Stochastic resonance

To discuss the effect of the periodic forcing in equation (2) we can follow straightforwardly the approach presented in Benzi *et al* (1981), hereafter referred to as BSV. Let us consider the two stochastic differential equations

$$\partial_t \phi = m\phi - \phi^3 + A\partial_x^2 \phi + B(x) + \sqrt{\varepsilon}\eta(x, t) \tag{17}$$

$$\partial_t \phi = m\phi - \phi^3 + A\partial_x^2 \phi - B(x) + \sqrt{\varepsilon}\eta(x, t). \tag{18}$$

We shall indicate by $\phi'_\pm$ and $\phi''_\pm$ the two stable steady solutions of equations (17) and (18) respectively. Equations (6) are modified into the two following equations:

$$m\phi' - \phi'^3 + A\partial_x^2 \phi' + B(x) = 0 \qquad \partial_x \phi' = 0 \quad \text{for } x = 0, L, \tag{19}$$

$$m\phi'' - \phi''^3 + A\partial_x^2 \phi'' - B(x) = 0 \qquad \partial_x \phi'' = 0 \quad \text{for } x = 0, L. \tag{20}$$

We assume $B(x) \ll 1$ uniformly in x and we look for the solutions of (19) and (20) in power series of $B(x)$:

$$\phi' = \Sigma_n B(x)^n \chi'_n. \tag{21}$$

The equations for χ_0' and χ_1' are:

$$m\chi_0' - \chi_0'^3 + A\partial_x^2\chi_0' = 0 \qquad \partial_x\chi_0' = 0 \quad \text{for } x = 0, L, \tag{22}$$

$$m\chi_1' B(x) - 3\chi_0'^2\chi_1' B(x) + A\partial_x^2(\chi_1' B(x)) + B(x) = 0 \tag{23}$$

$$\partial_x\chi_1' = 0 \quad \text{for } x = 0, L.$$

Analogous expressions hold for χ_0'' and χ_1''. The solutions of (22) correspond to the steady state solutions of equation (1). In order to compute the effect of the periodic forcing we need to evaluate, in analogy with BSV, the average exit times τ_+', τ_+'' from the basin of attraction of ϕ_+' and ϕ_+'' respectively. For small value of $B(x)$ we can estimate τ_+' and τ_+'' up to the first order in $B(x)$. To this purpose we can compute up to order $B(x)$ the potential V' and V'' of the steady states $\chi_0' + B(x)\chi_1'$ and $\chi_0'' + B(x)\chi_1''$. By V' and V'' we define the quantities

$$V' = \int \mathrm{d}x[\phi'^4/4 - m\phi'^2/2 + A(\partial_x\phi')^2 - B(x)\phi'],$$

$$V'' = \int \mathrm{d}x[\phi''^4/4 - m\phi''^2/2 + A(\partial_x\phi'')^2 + B(x)\phi'].$$

After some calculations we obtain:

$$V'[\chi_0' + B(x)\chi_1'] = V(\chi_0') - \int \mathrm{d}x\, B(x)\chi_0' \tag{24}$$

$$V''[\chi_0'' + B(x)\chi_1''] = V(\chi_0'') + \int \mathrm{d}x\, B(x)\chi_0'' \tag{25}$$

where $V(\chi_0')$ is the potential given in (4) and computed for $\phi = \chi_0'$. Because $\chi_0'' = \chi_0'$ are the steady states of equation (1) we can easily use equations (24) and (25) to estimate τ_+' and τ_+'':

$$\tau_+' \simeq C \exp 2\left[V(\phi^{(1)}) - V(\phi_+) - \int \mathrm{d}x\, B(x)\phi^{(1)} + \int \mathrm{d}x\, B(x)\phi_+\right]/\varepsilon \tag{26}$$

$$\tau_+'' \simeq C \exp 2\left[V(\phi^{(1)}) - V(\phi_+) + \int \mathrm{d}x\, B(x)\phi^{(1)} - \int \mathrm{d}x\, B(x)\phi_+\right]/\varepsilon. \tag{27}$$

Equations (26) and (27) can be used to discuss the statistical properties of equation (2). In this paper we discuss the case $B(x) = \text{constant}$. Then because $\int \mathrm{d}x\, B(x)\phi^{(1)} = 0$ it follows that

$$\tau_+' \simeq C \exp 2[V(\phi^{(1)}) - V(\phi_+) - BLm^{1/2}]/\varepsilon \tag{28}$$

$$\tau_+'' \simeq C \exp 2[V(\phi^{(1)}) - V(\phi_+) + BLm^{1/2}]/\varepsilon. \tag{29}$$

Following BSV we know that if

$$\tau_+'' \geqslant \pi/\omega \qquad \text{and} \qquad \tau_+' \ll \pi/\omega \tag{30}$$

then the average exit time of equation (2) is of order π/ω with variance of order τ_+'. In this case the solution of equation (2) is nearly periodic with period equal to $2\pi/\omega$.

Inequalities (30) can be used to compute the upper and lower limit (ε_{up}, ε_{low}) in the variance of the noise in order to obtain the mechanism of stochastic resonance

$$\varepsilon_{\text{low}} = 2(\Delta V + LBm^{1/2})/\ln(\pi/C\omega), \qquad \varepsilon_{\text{up}} = 2(\Delta V - LBm^{1/2})/\ln(\pi/C\omega).$$

We have investigated numerically the validity of the above results for the case $L=1$, $m=0.25$, $\omega=0.001\pi/3$ and for different values of ε. Numerical integration have been performed for $N=20$, 30 and 40 and no relevant changes have been observed. In figure 2 we plot the quantity $\int \phi(x)\,dx/L$ against time for different values of ε. It is clearly seen that the signal is periodic with period equal to $2\pi/\omega$ for ε around $\frac{1}{2}(\varepsilon_{up}+\varepsilon_{low})$. It is clearly seen that the solution of the signal is periodic with period nearly equal to $2\pi/\omega$.

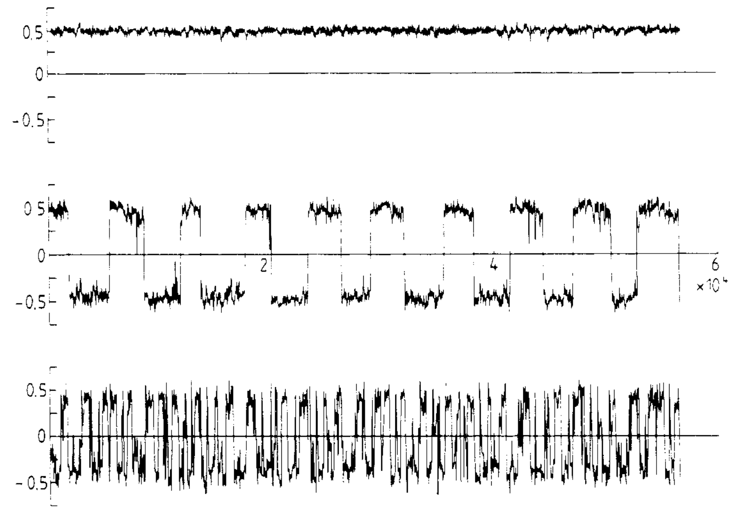

Figure 2. Plot of $\int \phi(x)\,dx/L$ against time for $\varepsilon/\Delta x=0.002$, $\varepsilon/\Delta x=0.07$ and $\varepsilon/\Delta x=0.11$. The parameter values of the numerical computation are $L=1$, $N=20$, $m=0.25$, $A=0.0125$, $\omega=0.001\pi/3$ and $B=\frac{1}{64}$.

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Appendix

In this appendix we compute explicitly the factor C_N defined in § 2.

Let z_i, $i=1\ldots N$ satisfy the set of stochastic differential equations:

$$dz_i=-(\partial U/\partial z_i)\,dt+\varepsilon^{1/2}\,dW_i. \tag{A1}$$

Let $P\equiv(z_1^s\ldots z_N^s)$ a stable fixed point of the deterministic part of equation (A1) and Ω the basin of attraction of P. The exit time from Ω is defined as

$$\tau(y_i)\equiv\inf[t: z_i(t)\in\partial\Omega,\ z_i(0)=y_i\in\Omega]$$

where $\partial\Omega$ is the boundary of Ω. The average value $\langle\tau(y)\rangle$ of $\tau(y)$ satisfies the differential equation (Dynkin 1965)

$$\tfrac{1}{2}\varepsilon\Sigma_i\partial^2_{ii}\langle\tau(y)\rangle - \Sigma_i\partial_i U\partial_i\langle\tau(y)\rangle = -1, \tag{A2}$$

with boundary conditions

$$\langle\tau(y)\rangle = 0 \qquad \text{for } y \in \partial\Omega. \tag{A3}$$

The solution of the equation (A2) can be computed analytically using the ray method reviewed by Ludwig (1975) and Shuss (1980). It turns out that $\langle\tau(y)\rangle$ is almost constant in Ω with a narrow boundary layer of thickness ε near $\partial\Omega$ which matches (A3). Using the saddle point technique it is possible to estimate $\langle\tau(y)\rangle$ inside Ω. The final result is

$$\langle\tau(y)\rangle \simeq \pi D^{-1/2}(P)\exp(2\Delta U/\varepsilon)/\Sigma_i(\partial U/\partial\gamma)^{1/2}H^{-1/2}(\zeta_i) \tag{A4}$$

where

$$\Delta U = \inf_{i=1\ldots n}(U(P_i^*) - U(P));$$

P_i^* are points of $\partial\Omega$ for which U is minimum;

$D(P) = \det \partial U/\partial y_i\,\partial y_j|_{y_i\equiv P}$

γ: outer normal versor to $\partial\Omega$ computed in P^*

ζ_i: local coordinate in $\partial\Omega$ orthogonal to γ

$H^{-1/2}(\zeta_i) = \det \partial^2 U/\partial\zeta_i\,\partial\zeta_j \qquad i, j = 1 \ldots N-1.$

Expression (A4) can also be written in the following way

$$\langle\tau(y)\rangle \simeq \pi\exp(2\Delta U/\varepsilon)/\{[|\Pi_i\lambda_i^s|]^{1/2}\Sigma_r\Lambda_r^{un}/[|\Pi_i\lambda_{r,i}^{un}|]^{1/2}\}$$

where λ_i^s are the eigenvalues of the matrix:

$$L_{ij} = \partial^2 U/\partial y_i\,\partial y_j \qquad i, j = 1 \ldots N,$$

computed in P; $\lambda_{r,i}^{un}$ are the eigenvalues of L_{ij} computed in P_r^* and $\Lambda_r^{un} = \max_i(\lambda_{r,i}^{un}) > 0$. In the case of equations (12) $n = 2$ and $\Lambda_1^{un} = \Lambda_2^{un}$ and $P_{1,2}^*$ are the instanton solutions $\pm\phi^{(1)}$. It follows that

$$\langle\tau(y)\rangle \simeq (\pi/2\Lambda_1^{un})[|\Pi\lambda_{1,i}^{un}/\Pi\lambda_i^s|]^{1/2}\exp(2\Delta V_N/\varepsilon). \tag{A5}$$

Then C_N is given by

$$C_N = (\pi/2\Lambda_1^{un})[|\Pi\lambda_{1,i}^{un}/\Pi\lambda_i^s|]^{1/2}. \tag{A6}$$

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